SPECTRAL SEQUENCES AND FROBENIUS GROUPS(1)

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Introduction. We analyze Frobenius groups by means of the cohomology theory of permutation representations, developed in [1] (square brackets refer to the references). The part of [1] which is needed for the present paper is exposed in Chapter 1 which makes the present paper, except for some proofs needed for this chapter, self-contained. Readers who know [1] can begin the present paper with Chapter 2.

CHAPTER 1. COHOMOLOGY OF PERMUTATION REPRESENTATIONS

1. **Permutation representations.** A permutation representation (G, X) of a group G consists of a nonempty set X on which the group acts on the left. That is, $\sigma x \in X$ for all $\sigma \in G$ and $x \in X$, and (1) $(\rho\sigma)x = \rho(\sigma x)$ for $\rho, \sigma \in G$ and $x \in X$; (2) $1x = x$ for $x \in X$. The unit element of G is denoted by 1. *Throughout this paper the symbol* (G, X) *stands for a fixed permutation representation where* G *and* X *are finite.*

Let $T_1, \cdots, T_u$ be the domains of transitivity of (G, X). (See [3, Chapter X], for domains of transitivity.) If $x_i \in T_i$ for $i = 1, \cdots, u$, we refer to the set $x_1, \cdots, x_u$ as *a set of representatives of the permutation representation* (G, X). For each $x \in X$, the subgroup of G whose elements leave x fixed is denoted by $H(x)$; i.e., $H(x) = \{\sigma \mid \sigma \in G,\ \sigma x = x\}$. The greatest common divisor of the indices $[G: H(x)]$ is called *the index of the permutation representation* (G, X). If $x_1, \cdots, x_u$ is a set of representatives of (G, X), the study of (G, X) is equivalent to the simultaneous study of the classes of conjugate subgroups of G to which $H(x_1), \cdots, H(x_u)$ belong.

2. **The standard complex of a permutation representation.** Z stands for the ring of the rational integers. The elements of X are denoted by $x_1, \cdots, x_m$ and the direct sum $Zx_1 \oplus \cdots \oplus Zx_m$ of the additive group of Z with itself m times by $Z[X]$. *We regard* $Z[X]$ *as a* G*-module* which we may do since G acts on the Z-base X of $Z[X]$.

G also acts on the Cartesian product X^r of X with itself r times $(r \geqq 1)$ by the rule $\sigma(x_1, \cdots, x_r) = (\sigma x_1, \cdots, \sigma x_r)$. The permutation representation (G, X^r) and the associated G-module $Z[X^r]$ are of course well defined.

The standard complex of the permutation representation (G, X) is defined as the augmented, acyclic G-complex $(C_r, \partial_r; r \in Z)$

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$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \xrightarrow{\partial_{-2}} C_{-3} \to \cdots$$

$$\epsilon \searrow \quad \nearrow \mu$$

$$Z$$

$$\nearrow \quad \searrow$$

$$0 \qquad 0$$

Here, Z is regarded as a G-module with trivial action and furthermore:

(1) $C_r = Z[X^{r+1}]$ for $r \geqq 0$.

(2) $C_{-r} = Z[X^r]$ for $r \geqq 1$.

(3) $\partial_r(x_1, \cdots, x_{r+1}) = \sum_{j=1}^{r+1} (-1)^{j+1}(x_1, \cdots, \hat{x}_j, \cdots, x_{r+1})$ for $r \geqq 1$; $\hat{x}_j$ indicates that x_j has to be omitted.

(4) $\partial_0(x) = x_1 + \cdots + x_m$ for all $x \in X$.

(5) $\partial_{-r}(x_1, \cdots, x_r) = \sum_{i=1}^{m}(x_i, x_1, \cdots, x_r) - \sum_{i=1}^{m}(x_1, x_i, x_2, \cdots, x_r) + \cdots (-1)^r \sum_{i=1}^{m}(x_1, \cdots, x_r, x_i)$ for $r \geqq 1$.

(6) $\epsilon(x) = 1$ for all $x \in X$.

(7) $\mu(1) = x_1 + \cdots + x_m$.

We refer to §1 of [1] for the proof that $(C_r, \partial_r; r \in Z)$ is indeed an augmented, acyclic G-complex. We denote this complex by $C.(X; G)$, where the lower dot reminds us that we are dealing with a chain complex.

3. **The cohomology groups of a permutation representation.** If A is a G-module, we denote the Z-complex $\mathrm{Hom}_G(C.(X; G), A)$ by $C^{\cdot}(X; G, A)$, where the upper dot reminds us that we are dealing with a cochain complex. *The rth cohomology group* $H^r(X; G, A)$ *of* A *relative to the permutation representation* (G, X) *is defined as the rth cohomology group of the complex* $C(X; G, A)$, *for all* $r \in Z$. (See Definition 2.1 of [1].)

We refer to §2 of [1] for the fact that the groups $H^r(X; G, A)$ generalize, in an obvious manner, the Adamson relative cohomology groups. We see from the same section that, if (G, X) is free of fixed points (that is, if $\sigma x \neq x$ for all $\sigma \neq 1$ and all $x \in X$), $H^r(X; G, A)$ may be identified with the customary cohomology group $H^r(G, A)$. (We do not tamper with the symbol $H^r(G, A)$; it retains its usual meaning.)

All the groups $H^r(X; G, A)$ are annihilated by the index d of (G, X) (see §1 for the index of a permutation representation); i.e., $dc = 0$ for $c \in H^r(X; G, A)$. This is the content of Corollary 10.2 of [1].

4. **The restriction mapping and corestriction mapping.** Let K be a subgroup of G and A a G-module. Since the group K also acts on the set X and A can be regarded as a K-module, the permutation representation (K, X) and the cohomology groups $H^r(X; K, A)$ are well defined. For all $r \in Z$, the restriction mapping is a homomorphism res: $H^r(X; G, A) \to H^r(X; K, A)$, and the corestriction mapping is a homomorphism cor: $H^r(X; K, A) \to H^r(X; G, A)$, and cor res consists of multiplying the elements of

$H^r(X; G, A)$ by the index $[G: K]$. (See §§8, 9, 10 of [1].) We need these facts only in the case where G, and hence K, acts without fixed points on X, in which case all of this is well known from the customary cohomology theory of groups. (For reasons explained in §9 of [1] we use the term "corestriction" instead of "transfer" or "Verlagerung.")

5. **Morphisms of permutation representations and the lift mapping.** We denote the positive half of the complex $C.(X; G)$, that is the G-complex $\cdots \to C_2 \to C_1 \to C_0 \to 0$, by C^+. If A is a G-module, the Z-complex $\mathrm{Hom}_G(C^+, A)$ has $H^r(X; G, A)$ as its rth cohomology group for $r \geqq 1$, while its zeroth cohomology group is the module A^G. (See Proposition 4.1 of [1].) We refer to A^G as the *unreduced* zeroth cohomology group of A.

Let now (L, Y) be a second permutation representation, where L is some finite group and Y is some finite set. The following definition occurs in the introduction of [1].

DEFINITION 5.1. *A morphism* θ: $(G, X) \to (L, Y)$ *from the permutation representation* (G, X) *to the permutation representation* (L, Y) *is a pair* (ϕ, f), *where* ϕ: $G \to L$ *is a group homomorphism and* f: $X \to Y$ *is a function; and where* ϕ *and* f *are interrelated by the condition that* $f(\sigma x) = \phi(\sigma) f(x)$ *for all* $\sigma \in G$ *and* $x \in X$.

We denote the L-module $Z[Y^{r+1}]$ by D_r for $r \geqq 0$, and denote the positive half $\cdots \to D_2 \to D_1 \to D_0 \to 0$ of the standard complex of (L, Y) by D^+. It is immediate that the morphism $\theta = (\phi, f)$ induces a chain mapping α: $C^+ \to D^+$, where α_r: $Z[X^{r+1}] \to Z[Y^{r+1}]$ is defined by $\alpha_r(x_1, \cdots, x_{r+1}) = (f(x_1), \cdots, f(x_{r+1}))$ for $r \geqq 0$. The mapping α is compatible, not only with the differentiations of C^+ and D^+, but also with the augmentation mappings $C_0 \to^{\epsilon} Z$ and $D_0 \to^{\epsilon'} Z$; that is, $\epsilon' \alpha_0 = \epsilon$. Furthermore, if we consider D^+ as a G-complex by means of the homomorphism ϕ: $G \to L$, α is a mapping of G-complexes.

Suppose that we are furthermore given a G-module A and an L-module B, together with a group homomorphism h: $B \to A$ which has the property that $h(\phi(\sigma) b) = \sigma h(b)$ for all $\sigma \in G$ and $b \in B$. There results a mapping of complexes β: $\mathrm{Hom}_L(D^+, B) \to \mathrm{Hom}_G(C^+, A)$, where β^r: $\mathrm{Hom}_L(Z[Y^{r+1}], B) \to \mathrm{Hom}_G(Z[X^{r+1}], A)$ is given by: If $c \in \mathrm{Hom}_L(Z[Y^{r+1}], B)$ then $(\beta^r c)(x_1, \cdots, x_{r+1}) = h(c(f(x_1), \cdots, f(x_{r+1})))$. The mapping β induces homomorphisms θ^r: $H^r(Y; L, B) \to H^r(X; G, A)$ for $r \geqq 0$, where zero[th] cohomology groups are to be considered as unreduced.

If ϕ is a monomorphism and f is $(1, 1)$ and onto and h: $A \to A$ is the identity mapping, θ^r may be interpreted as the restriction mapping discussed in §4. In order to obtain the lift mapping, we assume that ϕ is an epimorphism and denote $\ker(\phi) = N$. We observe that then the G-module A gives rise to the L-module $B = A^N$ and that the inclusion mapping

$A^N \to A$ may be used as the h: $B \to A$. The following definition is Definition 13.1 of [1].

DEFINITION 5.2. *Let $\theta = (\phi, f)$: $(G, X) \to (L, Y)$ be a morphism of permutation representations, where both ϕ and f are epimorphisms. Let A be a G-module and h the inclusion mapping of A^N into A, where $N = \ker(\phi)$. The induced homomorphism θ^r: $H^r(Y; L, A^N) \to H^r(X; G, A)$ is called the inflation mapping or lift mapping, for $r \geqq 0$. Zeroth cohomology groups are to be considered as unreduced.*

See §13 of [1] for the homomorphisms induced by θ in the cohomology groups of negative dimensions.

6. **The spectral sequence interrelating $H^r(G, A)$ and $H^r(X; G, A)$.** The positive half C^+ of the standard complex of the permutation representation (G, X) is a G-complex. Consequently, for every G-module A, the Z-complex $\mathrm{Ext}^0_G(C^+, A) = 0 \to \mathrm{Ext}^0_G(C_0, A) \to \mathrm{Ext}^0_G(C_1, A) \to \cdots$ is well defined for $q \geqq 0$. Its cohomology groups $H^p[\mathrm{Ext}^q_G(C^+, A)]$, for $p, q \geqq 0$, determine the initial term of the following spectral sequence which interrelates the groups $H^r(G, A)$ and $H^r(X; G, A)$ for $r \geqq 0$. See §12 of [1] for the derivation of this spectral sequence.

THEOREM 6.1. *To each G-module A is associated a spectral sequence which has the following components:*

(1) *The final term $E^p(A) = H^p(G, A)$ for $p \geqq 0$. Here, $H^0(G, A)$ is unreduced, i.e., $E^0(A) = A^G$.*

(2) *The initial term $E_2^{p,q}(A) = H^p[\mathrm{Ext}^q_G(C^+, A)]$ for $p, q \geqq 0$.*

(3) *$E_2^{p,0}(A) = H^p(X; G, A)$ for $p \geqq 0$. Here, $H^0(X; G, A)$ is unreduced, i.e., $E_2^{0,0}(A) = A^G$.*

In order to analyze the groups $\mathrm{Ext}^q_G(C_p, A)$, we select $p, q \geqq 0$ and denote the domains of transitivity of the permutation representation (G, X^{p+1}) by $T_1, \cdots, T_u$. Then, $C_p = Z[X^{p+1}] = Z[T_1] \oplus \cdots \oplus Z[T_u]$, where $\oplus$ denotes the direct sum of G-modules. Consequently, $\mathrm{Ext}^q_G(C_p, A)$ is the direct sum of the groups $\mathrm{Ext}^q_G(Z[T_i], A)$, $i = 1, \cdots, u$. Each group $\mathrm{Ext}^q_G(Z[T_i], A)$ is of the type discussed in the following proposition. (For the proof, see Proposition 12.1 of [1].)

PROPOSITION 6.1. *Let (G, T) be a transitive permutation representation. Select $t \in T$. Then, for every G-module A, $\mathrm{Ext}^q_G(Z[T], A) \cong H^q(H(t), A)$, $q \geqq 0$. Here, $H^0(H(t), A)$ has to be considered as unreduced.*

We now select $e_i \in T_i$ for $i = 1, \cdots, u$, i.e., $e_1, \cdots, e_u$ is a set of representatives of (G, X^{p+1}). (See §1 for the notion of a set of representatives of a permutation representation.) Each e_i is a sequence $(x_1, \cdots, x_{p+1})$ of $p + 1$ elements of X and the subgroup $H(e_i) = \{\sigma \mid \sigma \in G, \sigma e_i = e_i\}$ of G is

equal to $H(x_1) \cap \cdots \cap H(x_{p+1})$. (See §1 for the definition of $H(x)$ where $x \in X$.) The following corollary of Proposition 6.1 describes the groups $\mathrm{Ext}_G^q(C_p, A)$.

COROLLARY 6.1. *Let $p, q \geqq 0$ and let $e_1, \cdots, e_u$ be a set of representatives of (G, X^{p+1}). Then, for each G-module A, $\mathrm{Ext}_G^q(C_p, A)$ is equal to the direct sum of the groups $H^q(H(e_i), A)$, $i = 1, \cdots, u$. The groups $H^0(H(e_i), A)$ must be considered as unreduced.*

If we apply Corollary 6.1 in the case that $p = 0$ and use that $E_2^{0,q}(A) \subset \mathrm{Ext}_G^q(C_0, A)$, we find:

COROLLARY 6.2. *Let $x_1, \cdots, x_u$ be a set of representatives of the permutation representation (G, X). For each G-module A and $q \geqq 0$, $E_2^{0,q}(A)$ is a subgroup of the direct sum of the groups $H^q(H(x_i), A)$, $i = 1, \cdots, u$.*

7. **The mappings built into the spectral sequence of Theorem 6.1.** Every spectral sequence $\{E_r^{p,q}, E^p; p, q \geqq 0, r \geqq 2\}$ has homomorphisms $E_2^{p,0} \to E^p \to E_2^{0,p}$, for $p \geqq 0$, built into it. Hence, associated with the spectral sequence of Theorem 6.1, there are homomorphisms

$$H^p(X; G, A) \xrightarrow{\lambda^p} H^p(G, A) \xrightarrow{\rho_0^p} E_2^{0,p}$$

for $p \geqq 0$. In order to get rid of the term $E_2^{0,p}$, we select a set of representatives $x_1, \cdots, x_u$ of (G, X) and conclude from Corollary 6.2 that we have mappings

$$H^p(X; G, A) \xrightarrow{\lambda^p} H^p(G, A) \xrightarrow{\rho^p} \sum_{i=1}^{u} H^p(H(x_i), A)$$

for $p \geqq 0$; here $\sum$ denotes the direct sum of abelian groups and ρ^p is equal to ρ_0^p followed by the inclusion mapping. *The theory of spectral sequences tells us that $\rho^p \lambda^p = 0$ for $p \geqq 1$.*

Interpretation of ρ^p. Consider the restriction mapping res_i: $H^p(G, A) \to H^p(H(x_i), A)$ for $i = 1, \cdots, u$. If $c \in H^p(G, A)$, $\rho^p(c) = \sum_{i=1}^{u} \mathrm{res}_i(c)$. (See §14 of [1].) Observe that, if (G, X) is transitive, ρ^p is simply the restriction mapping res: $H^p(G, A) \to H^p(H(x), A)$ where $x \in X$.

Interpretation of λ^p. As above, $x_1, \cdots, x_u$ is a set of representatives of (G, X). Consider the permutation representation (G, W), where G acts on W without fixed points and where (G, W) has u domains of transitivity $T_1, \cdots, T_u$. We view each T_i as G and the action of G on T_i as the left multiplication of G. There then exists an obvious morphism of permutation representations $\theta = (\phi, f)$: $(G, W) \to (G, X)$ which is defined by: ϕ is the identity mapping of G; if $t \in T_i$, $f(t) = tx_i$ for $i = 1, \cdots, u$. Clearly, ϕ and f are epimorphisms, and $\ker(\phi) = 1$ and $H^p(W; G, A) \cong H^p(G, A)$. We conclude from Definition 5.2 that the lift mapping θ^p

maps $H^p(X; G, A)$ into $H^p(G, A)$, for $p \geqq 0$. The homomorphism λ^p may be identified with this lift mapping θ^p. (See §14 of [1].)

In applications (see for instance §10), it will usually be necessary to investigate also the mappings $E_{p+1}^{0,p} \to E_{p+1}^{p+1,0}$ which are built into every spectral sequence. Since $E_{p+1}^{0,p}$ is a subgroup of $E_2^{0,p}$, and $E_{p+1}^{p+1,0}$ is a quotient group of $E_2^{p+1,0}$, these mappings, when interpreted in terms of the components of the spectral sequence of Theorem 6.1, become the "transgressions."

We refer the reader to [1] for the exact sequences with five terms which are built into the spectral sequence of Theorem 6.1, and also for the comparison of this spectral sequence with the one of Serre-Hochschild. Neither of these two items is needed for the present paper.

Chapter 2. Frobenius Groups

8. **Frobenius representations.** *For the remainder of this paper we assume that the permutation representation (G, X) is transitive and has the property that, if $\sigma \in G$ and σ has at least two fixed points in X, then $\sigma = 1$.* We call such a permutation representation (as in §7 of [1]) *a Frobenius representation,* and we shall analyze our fixed, Frobenius representation (G, X) by studying its spectral sequence. (See §10.)

For the sake of proof arrangement, we found it convenient not to exclude the two trivial cases: (1) (G, X) is the regular representation of G; (2) X consists of only one element. Every group admits of course these two trivial Frobenius representations. If a group admits a nontrivial Frobenius representation, it is commonly called a Frobenius group. (See for instance [2].)

9. **Well-known facts about Frobenius representations.** We fix, *once and for all,* an element $x_0 \in X$ and denote the subgroup $H(x_0)$ of G by H, i.e., $H = \{\sigma \mid \sigma \in G, \sigma x_0 = x_0\}$. We denote the order of H by h, the order of G by n and the index $[G: H]$ by m; consequently, m is the number of elements of X and $n = hm$. (See [3, Chapter X], for the classical theory of permutation representations.) It is clear that the group H acts on the set $X - \{x_0\}$, and that the permutation representation $(H, X - \{x_0\})$ is free of fixed points. Hence, if w is the number of domains of transitivity of $(H, X - \{x_0\})$, then $wh = m - 1$.

10. **The spectral sequence interrelating $H^r(G, A)$ and $H^r(X; G, A)$.** Let A be a G-module. The spectral sequence which interrelates the cohomology groups $H^r(G, A)$ and $H^r(X; G, A)$ for $r \geqq 0$ (zeroth cohomology groups are unreduced all through this chapter), has as initial term $E_2^{p,q}(A) = H^p[\mathrm{Ext}_G^q(C^+, A)]$ where $p, q \geqq 0$; see Theorem 6.1. In order to compute the groups $\mathrm{Ext}_G^q(C_p, A)$ for some fixed $p, q \geqq 0$, we have to choose a set of representatives $e_1, \cdots, e_u$ of the permutation representation (G, X^{p+1}) and to determine the groups $H^q(H(e_i), A)$, where $i = 1, \cdots, u$. (See Corollary

6.1.) We remember that $e_i = (x_1, \cdots, x_{p+1})$ and that $H(e_i) = H(x_1) \cap \cdots \cap H(x_{p+1})$ for appropriate $x_1, \cdots, x_{p+1} \in X$.

The subset of X^{p+1} which consists of those $(p+1)$-tuples $(x, \cdots, x)$ all of whose coordinates are equal to the same element $x \in X$, is called *the diagonal of* X^{p+1} and will be denoted by Δ_{p+1}. We observe that Δ_{p+1} is a domain of transitivity of the permutation representation (G, X^{p+1}) and we shall always choose $e_1 = (x_0, \cdots, x_0) \in \Delta_{p+1}$.

PROPOSITION 10.1. *$H(e_1) = H$ and $H(e_i) = 1$ for $i = 2, \cdots, u$. Consequently, $\mathrm{Ext}^q_G(C_p, A) = \mathrm{Ext}^q_G(Z[\Delta_{p+1}], A) = H^q(H, A)$ for $q \geqq 1$ and $p \geqq 0$.*

Proof. It is trivial that $H(e_1) = H$. If $1 < i \leqq u$, at least two of the coordinates of $e_i = (x_1, \cdots, x_{p+1})$ are distinct from one another. Consequently, *since (G, X) is a Frobenius representation, $H(e_i) = 1$* and hence $H^q(H(e_i), A) = 0$ for $q \geqq 1$ and $i = 2, \cdots, u$. We enumerate the domains of transitivity $T_1 = \Delta_{p+1}, T_2, \cdots, T_u$ of the permutation representation (G, X^{p+1}) in such a way that $e_i \in T_i$ for $i = 1, \cdots, u$. Then, $\mathrm{Ext}^q_G(C_p, A) = \mathrm{Ext}^q_G(Z[\Delta_{p+1}], A) \oplus \sum_{i=2}^{u} \mathrm{Ext}^q_G(Z[T_i], A)$ for $q \geqq 0$ where $\oplus$ and $\sum$ denote the direct sum of abelian groups. (See §6.) We conclude from Proposition 6.1 that $\mathrm{Ext}^q_G(Z[\Delta_{p+1}], A) = H^q(H, A)$ and that $\mathrm{Ext}^q_G(Z[T_i], A) = H^q(H(e_i), A)$ for $q \geqq 0$ and $i = 2, \cdots, u$. Proposition 10.1 now follows from our previous remarks about $H^q(H(e_i), A)$ for $q \geqq 1$ and $i = 2, \cdots, u$. Done.

We conclude from Proposition 10.1 that, for $p \geqq 0$ and $q \geqq 1$, the initial term $E_2^{p,q}(A)$ is the pth cohomology group of the complex

$$0 \to \mathrm{Ext}^q_G(Z[\Delta_1], A) \xrightarrow{\delta_0} \mathrm{Ext}^q_G(Z[\Delta_2], A) \xrightarrow{\delta_1} \cdots .$$

Observe that all cochain groups of this complex are equal to $H^q(H, A)$. We must now determine the differential operator δ_p: $\mathrm{Ext}^q_G(Z[\Delta_{p+1}], A) \to \mathrm{Ext}^q_G(Z[\Delta_{p+2}], A)$ of this complex.

PROPOSITION 10.2. *δ_p is the zero mapping if p is even, and δ_p is an isomorphism if p is odd; $p \geqq 0$.*

Proof. We know from §§2 and 6 that, in the notation of homological algebra, $\delta_p = \mathrm{Ext}^q_G(\partial'_{p+1}, 1_A)$, where ∂'_{p+1} is the restriction of the chain mapping ∂_{p+1}: $C_{p+1} \to C_p$ to $Z[\Delta_{p+2}]$. (See §2 for the definition of ∂_{p+1}.) Select some $(x, \cdots, x) \in \Delta_{p+2}$, i.e., $x \in X$ and $(x, \cdots, x)$ has $p+2$ coordinates. If p is even, obviously $\partial_{p+1}(x, \cdots, x) = 0$, whence $\partial'_{p+1} = 0$. If p is odd, obviously $\partial'_{p+1}(x, \cdots, x) = (x, \cdots, x)' \in \Delta_{p+1}$ where $(x, \cdots, x)'$ has one coordinate less than $(x, \cdots, x)$. It follows that ∂'_{p+1}: $Z[\Delta_{p+2}] \to Z[\Delta_{p+1}]$ is an isomorphism when p is odd. Wonderful homological algebra now tells us that $\mathrm{Ext}^q_G(\partial'_{p+1}, 1_A) = 0$ (an isomorphism) when p is even (odd). Done.

THEOREM 10.1. *The spectral sequence which interrelates* $H^r(G,A)$ *and* $H^r(X;G,A)$ *has the following components*:

(1) *The final term* $E^p(A) = H^p(G,A)$ *for* $p \geqq 0$.

(2) *The initial term* $E_2^{p,q}(A) = 0$ *if both* $p,q > 0$.

(3) $E_2^{p,0}(A) = H^p(X;G,A)$ *for* $p \geqq 0$.

(4) $E_2^{0,q}(A) = H^q(H,A)$ *for* $q \geqq 1$.

The transgression mapping t^p: $E_2^{0,p}(A) \to E_2^{p+1,0}(A)$ *is the zero mapping for* $p \geqq 1$.

Proof. The spectral sequence in question is the one of Theorem 6.1. Hence (1) and (3) above state the same as (1) and (3) of the quoted theorem. If $q > 0$, $E_2^{p,q}(A)$ is the pth cohomology group of the complex discussed in Proposition 10.2. It follows trivially from Proposition 10.2 that the zeroth cohomology group of this complex is equal to its zeroth cochain group $H^q(H,A)$; and that all other cohomology groups of this complex vanish. This proves (4) and (2) above, and we now turn to the transgression mappings.

It follows from the general theory of spectral sequences that, since $E_2^{p,q} = 0$ when both $p,q > 0$ (we are leaving the "A" out), $E_2^{0,p} = E_{p+1}^{0,p}$ for $p \geqq 1$ and $E_2^{p,0} = E_p^{p,0}$ for $p \geqq 2$. The differentiation operator of the complex $\{E_{p+1}^{a,b}\}$ maps $E_{p+1}^{0,p}$ into $E_{p+1}^{p+1,0}$, i.e., maps $E_2^{0,p}$ into $E_2^{p+1,0}$ for $p \geqq 1$. The resulting mapping t^p: $H^p(H,A) \to H^{p+1}(X;G,A)$ is the transgression mapping mentioned in §7. We know from §3 that $hH^p(H,A) = 0$ for $p \geqq 1$ (zeroth cohomology groups are unreduced) and that $mH^{p+1}(X;G,A) = 0$ for $p \geqq 0$. Since $(m,h) = 1$ by the last equation of §9, $t^p = 0$ for $p \geqq 1$. Done.

11. **The lift-restriction sequence.** For each $p \geqq 1$, the sequence of maps $H^p(X;G,A) \to^{\lambda^p} H^p(G,A) \to^{\rho^p} H^p(H,A)$ is well defined (see §7). Here, λ^p is the lift mapping and ρ^p is the restriction mapping res^p.

THEOREM 11.1. *The sequence* $0 \to H^p(X;G,A) \to^{\lambda^p} H^p(G,A) \to^{\mathrm{res}^p} H^p(H,A) \to 0$ *is exact for* $p \geqq 1$.

Proof. Every spectral sequence $\{E_r^{p,q}, E^p; p,q \geqq 0, r \geqq 2\}$ which has the property that $E_2^{p,q} = 0$ when both p and q are positive, has built into it exact sequences $0 \to E_\infty^{p,0} \to E^p \to E_\infty^{0,p} \to 0$ for $p \geqq 1$. If furthermore the transgression mappings t^p: $E_2^{0,p} \to E_2^{p+1,0}$ are zero for $p \geqq 1$, we may identify $E_\infty^{0,p}$ with $E_2^{0,p}$, and $E_\infty^{p,0}$ with $E_2^{p,0}$. All these conditions are satisfied in the case of the spectral sequence of Theorem 10.1, and (1), (3) and (4) of that theorem show that the exact sequences $0 \to E_2^{p,0} \to E^p \to E_2^{0,p} \to 0$ are those mentioned in Theorem 11.1. Done.

If S is an abelian group and $k \in Z$ (Z stands for the ring of the rational integers), we denote by $S(k)$ the subgroup of S whose elements c have the

property that $kc = 0$. We use notations such as $H^p(G, A, k)$ instead of $(H^p(G, A))(k)$. The corestriction mapping cor^p: $H^p(H, A) \to H^p(G, A)$ (see §4) obviously maps $H^p(H, A)$ into $H^p(G, A, h)$, and we shall denote this mapping $H^p(H, A) \to H^p(G, A, h)$ also by cor^p. (Such abuse!)

THEOREM 11.2. *For all* $p \geqq 1$:

(1) $H^p(G, A) = H^p(G, A, m) \oplus H^p(G, A, h)$, *where* $\oplus$ *denotes the direct sum of abelian groups.*

(2) $H^p(G, A, m) \cong H^p(X; G, A)$; λ^p *maps* $H^p(X; G, A)$ *isomorphically onto* $H^p(G, A, m)$.

(3) $H^p(G, A, h) \cong H^p(H, A)$; res^p *maps* $H^p(G, A, h)$ *isomorphically onto* $H^p(H, A)$.

(4) *The inverse of* $\mathrm{res}^p | H^p(G, A, h)$: $H^p(G, A, h) \to H^p(H, A)$ *is* cor^p: $H^p(H, A) \to H^p(G, A, h)$. *Consequently,* cor^p *is an isomorphism and* $\mathrm{res}^p \mathrm{cor}^p$ *is the identity of* $H^p(H, A)$.

Proof. The lift-restriction sequence is an exact sequence of the form $0 \to R \to^i S \to^j T \to 0$ where R, S, T are abelian groups and i, j are homomorphisms; furthermore, $mR = 0$ and $hT = 0$ and $(m, h) = 1$. One shows immediately that then $S = S(m) \oplus S(h)$; and that i maps R isomorphically onto $S(m)$ and that j maps $S(h)$ isomorphically onto T. This proves (1), (2) and (3), and we now turn to (4). The endomorphism $\mathrm{cor}^p \mathrm{res}^p$ of $H^p(G, A)$ consists of multiplying the elements of $H^p(G, A)$ by m, and we recall from §9 that $wh = m - 1$. Consequently, if $c \in H^p(G, A, h)$, then $(\mathrm{cor}^p \mathrm{res}^p)(c) = mc = (wh + 1)c = c$, which shows that cor^p is the inverse of $\mathrm{res}^p | H^p(G, A, h)$. Done.

One can, of course, investigate the cohomology theory of the Frobenius representation (G, X) also for negative dimensions. The lift-restriction sequence then becomes the corestriction-deflation sequence. (See Remark 13.1 of [1]; all arrows are inverted.) Instead, we turn to the applications of Theorems 11.1 and 11.2 to Frobenius groups.

REMARK 11.1. Theorems 11.1 and 11.2 are not new. D. K. Faddeev studies in [11] the lift-restriction sequence, assuming only that the order of the subgroup H of G is relatively prime to the index $[G:H]$. Under this assumption, his result is that the lift $H^q(X; G, A) \to H^q(G, A)$ sends $H^q(X; G, A)$ isomorphically onto the $[G:H]$-annihilated part of $H^q(G, A)$ and that the restriction map sends the $[H:1]$-annihilated part of $H^q(G, A)$ isomorphically onto the stable subgroup of $H^q(H, A)$. (For "stable," see [8, p. 257].) Faddeev's methods can be used to obtain Theorems 11.1 and 11.2. Conversely, Faddeev's results can be obtained by the spectral sequence technique of the present paper. Faddeev's paper is in Russian but has been reviewed in Math. Reviews **17**, p. 11. The author is indebted to the referee for having pointed out Faddeev's paper.

12. **The Frobenius theorem.** We shall denote by N the subset of G which consists of 1 together with those elements of G which have no fixed point in X. It is clear that, if $\pi \in N$, then $\pi^{-1} \in N$ and $\sigma\pi\sigma^{-1} \in N$ for all $\sigma \in G$. A famous theorem by Frobenius states that N is a group (see §247 of [3]), and we shall refer to this theorem as *the Frobenius theorem*. It is obvious that this group must be a normal subgroup of G.

It is the purpose of §§13 and 14 to prove the Frobenius theorem in the case that H is solvable and to identify N with the appropriate "higher commutator subgroup" of G. (See Theorem 14.1. §§15 and 18 contain a discussion of the nonsolvable case.) *Hence we consider N as a set* and must show that this set is closed under multiplication. We shall make use of the following proposition which is proved in §134 of [3].

PROPOSITION 12.1. *N contains m elements. These are precisely the elements σ of G which have the property that $\sigma^m = 1$. If an element σ of G does not belong to N, $\sigma^h = 1$.*

13. **The commutator subgroups of G and H.** We shall denote the commutator subgroup of G (of H) by G^1 (by H^1).

LEMMA 13.1. $H^1 = H \cap G^1$.

Proof. Consider the G-module Z with G acting trivially on Z. According to Theorem 11.2 (4), $\mathrm{cor}^2\colon H^2(H, Z) \to H^2(G, Z)$ is a monomorphism. As is well known from the fixed point free theory, this means that the natural mapping $\alpha\colon H/H^1 \to G/G^1$ is a monomorphism. This mapping is given by $\alpha(\rho H^1) = \rho G^1$ where $\rho \in H$, whence $\ker(\alpha) = (H \cap G^1)/H^1$. Since $\ker(\alpha) = 1$, we are done.

If $H = 1$, our Frobenius representation (G, X) is free of fixed points, whence then $N = G$. In that case it is of course not true that, necessarily, $N \subset G^1$ or that $G/G^1 \cong H/H^1$.

LEMMA 13.2. *If $H \neq 1$, $N \subset G^1$ and $G/G^1 \cong H/H^1$.*

Proof. *Case* 1. $H \neq H^1$. (Then, obviously, $H \neq 1$.) We return to the G-module Z with G acting trivially on it. We know from the fixed point free theory that $H^2(G, Z) \cong G/G^1$, whence we conclude from Proposition 12.1 that for all $a \in H^2(G, Z)$, either $ma = 0$ or $ha = 0$. Consequently, since $(m, h) = 1$, either $H^2(G, Z, m) = 0$ or $H^2(G, Z, h) = 0$. Theorem 11.2 (3) tells us that $H^2(G, Z, h) \cong H^2(H, Z)$ which implies, since $H^2(H, Z) \cong H/H^1$ and $H \neq H^1$, that $H^2(G, Z, h) \neq 0$. We conclude that $H^2(G, Z, m) = 0$ which, using the identification of $H^2(G, Z)$ and G/G^1, means the following: If $\sigma \in G$ and $\sigma^m \in G^1$, then $\sigma \in G^1$. Since, for all $\sigma \in N$, $\sigma^m = 1$, $N \subset G^1$. Finally, Theorem 11.2 tells us that $H^2(G, Z) = H^2(G, Z, h) \cong H^2(H, Z)$, whence $G/G^1 \cong H/H^1$.

Case 2. $H \neq 1$, *but otherwise* H *is arbitrary*. We delay the proof of this case until §17, because we are not able to handle Case 2 without the use of the Frobenius theorem. The proof of Case 2 is dependent on Case 1, and it would be of interest to have a proof of Case 2 which does not make use of the Frobenius theorem. We have to use Case 1 already in the next section.

14. **The Frobenius theorem for solvable** H. *We assume for this section that* H *is solvable and that* $H \neq 1$. We denote the commutator subgroup of H^1 by H^2; of H^2 by H^3, etc. We consequently have the properly descending sequence $H = H^0 \supset H^1 \supset H^2 \supset \cdots \supset H^e = 1$, where $e \geqq 1$. We use the same notation $G = G^0 \supset G^1 \supset G^2 \supset \cdots$ for the higher commutator subgroups of G.

THEOREM 14.1. $G^i/G^{i+1} \cong H^i/H^{i+1}$ *for* $i = 0, \cdots, e-1$. *Furthermore,* $G^e = N$.

Proof. We know from Case 1 of Lemma 13.2 that $G/G^1 \cong H/H^1$, and we now study the permutation representation (G^1, X). The subgroup $[N]$ of G which is generated by the elements of N (we want to prove that $[N] = N$) acts transitively on X; see Theorem XI, p. 86, of [4]. Since $[N] \subset G^1$ by Case 1 of Lemma 13.2, the permutation representation (G^1, X) is transitive. It is trivial that, if $\sigma \in G^1$ and σ has at least two fixed points in X, then $\sigma = 1$. We conclude that (G^1, X) is a Frobenius representation. The elements of G^1 which have no fixed point in X, together with 1, constitute again the set N. Returning to the element $x_0 \in X$, chosen in §1, we see that the group $\{\sigma \mid \sigma \in G^1, \sigma x_0 = x_0\}$ is equal to $G^1 \cap H$ and hence, by Lemma 13.1, to H^1. The upshot is that we changed from the original "Frobenius triple" (N, G, H) to the new Frobenius triple (N, G^1, H^1). If $H^1 = 1$, we stop. If $H^1 \neq 1$, we conclude again from Case 1 of Lemma 13.2 that $G^1/G^2 \cong H^1/H^2$, and then go over to the new Frobenius triple (N, G^2, H^2). In this way we show that $G^i/G^{i+1} \cong H^i/H^{i+1}$ for $i = 0, \cdots, e-1$ and reach the Frobenius triple (N, G^e, H^e). Since $H^e = 1$, $N = G^e$. Done.

REMARK 14.1. The investigation of the solvable case is not complete. We know from Theorem 1 of [5] that N is solvable, whence G^e and hence G is solvable. In particular, $G^e \neq G^{e+1}$, unless we are in the trivial case that $N = 1$ (and hence $H = G$). This situation is in marked contrast with the statement that $G^e = G^{e+1}$ if H is not solvable; see Theorem 18.1. Furthermore, it follows from §248 of [3] (we avoid the well-known error in that section) together with Theorem 11.6 on p. 262 of [8], that the cohomology of H is periodic, even if H is not solvable. It would be of great interest to obtain proofs of the solvability of G and the periodicity of H by means of the cohomological methods of this paper.

Other proofs that N is a group (under the assumption that H is solvable) have been given by Shaw in [6] and Grün in [7]. The theorems concerning the higher commutator subgroups of G and H seem to have been missed.

15. **The Frobenius theorem for nonsolvable H.** *We assume for this section that H is nonsolvable.* The most tempting proof that N is a group is now obtained by invoking the theorem of Feit and Thompson "groups of odd order are solvable." We conclude from it that H has even order and can then fall back on the completely elementary proof, given in §134 of [3], that N is an *abelian* group. As things stand at the moment, this proof arrangement is perhaps not quite fair because of the heavy use Feit and Thompson make of Frobenius groups in the proof of their theorem.

One can avoid the theorem of Feit and Thompson, if one proves first that the cohomology of H is periodic, as suggested in Remark 14.1. Namely, the periodicity of H tells us that all Sylow subgroups of H of odd order are cyclic. (See Theorem 11.6, p. 262 of [8].) Since H is nonsolvable, not all Sylow subgroups of H can be cyclic. (See Theorem 10, p. 145 of [9].) Consequently, H possesses a Sylow subgroup of even order (by necessity a generalized quaternion group), and we have again obtained that the order of H is even.

In the remainder of this paper we shall consider the Frobenius theorem as proved and use it to show that $H^2(X; G, Z) = 0$ (also if H is solvable). This will then enable us to compare the higher commutator subgroups of G and H also if H is nonsolvable. (See Theorem 18.1.)

16. **Identification of $H^p(X; G, A)$ and $H^p(N, A)^G$.** We drop all solvability conditions on H, and use that N is a subgroup of G.

LEMMA 16.1. *Let A be a G-module and $p \geqq 1$. Then, $H^p(H, A) \cong H^p(H, A^N)$ and $H^p(X; G, A) \cong H^p(N, A)^G$.*

Proof. It is immediate that we have the exact sequence $1 \to N \to G \to H \to 1$. Since $(m, h) = 1$, where m is the order of N and h is the order of H, the customary Serre-Hochschild spectral sequence gives rise to the exact sequence

$$0 \to H^p(H, A^N) \xrightarrow{\lambda^p} H^p(G, A) \xrightarrow{\text{res}^p} H^p(N, A)^G \to 0,$$

where λ^p is the inflation mapping and res^p is the restriction mapping. (See p. 127 of [10].) Since $hH^p(H, A^N) = 0$ and $mH^p(N, A)^G = 0$ we conclude, as in Theorem 11.2, that $H^p(G, A, h) \cong H^p(H, A^N)$ and $H^p(G, A, m) \cong H^p(N, A)^G$. Lemma 16.1 now follows from (2) and (3) of Theorem 11.2. Done.

COROLLARY 16.1. *$H^p(H, A/A^N) = 0$ for $p > 0$, and the unreduced group $H^0(H, A/A^N) \cong A^H/A^G$.*

Proof. The exact G-sequence $0 \to A^N \to^i A \to A/A^N \to 0$, where i is the inclusion mapping, can be regarded as an H-sequence. The resulting

exact cohomology sequence $0 \to H^0(H, A^N) \to H^0(H, A) \to H^0(H, A/A^N) \to H^1(H, A^N) \to \cdots$, together with the first isomorphism of Lemma 16.1, gives Corollary 16.1. Done.

In the next section, we will use the second isomorphism of Lemma 16.1 in the case that $p = 2$ and $A = Z$ with G acting trivially on Z. We see from Lemma 16.1 that $H^2(X; G, Z) \cong (N/N^1)^G$, where N^1 is the commutator subgroup of N. As follows from [10], the action of G on N/N^1 is the natural one: If $\sigma \in G$ and $\pi \in N$, $\sigma(\pi N^1) = \sigma\pi\sigma^{-1}N^1$. Consequently, $\pi N^1 \in (N/N^1)^G$ if and only if $\pi^{-1}\sigma\pi\sigma^{-1} \in N^1$ for all $\sigma \in G$.

17. **The vanishing of $H^2(X; G, Z)$.** We denote the center of G by C.

LEMMA 17.1. *Let $H \neq 1$ and $H \neq G$. Then, $C = 1$.*

Proof. Since $H \neq G$, there exists $\pi \in N$ where $\pi \neq 1$. If $\sigma \in G$ and $\sigma\pi = \pi\sigma$, every fixed point $x \in X$ of σ causes πx to be another fixed point of σ. Hence, either $\sigma = 1$ or σ has no fixed points, i.e., $\sigma \in N$. In particular, $C \subset N$. Suppose now that C contained an element $\sigma \neq 1$. Since $\sigma \in N$, the above reasoning shows that then $G = N$. This would imply that $H = 1$, which has been excluded. Done.

It is obvious that neither of the two conditions $H \neq 1$ or $H \neq G$ can be removed from Lemma 17.1.

THEOREM 17.1. *Let $H \neq 1$. Then, $H^2(X; G, Z) = 0$.*

Proof. *Case* 1. $H \neq H^1$. We proved under Case 1 of Lemma 13.2 that, if $H \neq H^1$, $H^2(G, Z, m) = 0$. According to (2) of Theorem 11.2 this means precisely that $H^2(X; G, Z) = 0$.

Case 2. $H = H^1$. Since $H \neq 1$, H is not solvable whence N is an abelian group. (See §15.) We see from the last paragraph of §16 that now $H^2(X; G, Z) \cong N^G$, and that an element π of N belongs to N^G if and only if $\pi^{-1}\sigma\pi\sigma^{-1} = 1$ for all $\sigma \in G$. This means of course that $H^2(X; G, Z) \cong N \cap C$. If $H = G$, $N = 1$ and hence $N \cap C = 1$; if $H \neq G$, $C = 1$ by Lemma 17.1 and again $N \cap C = 1$. Done.

Proof of Case 2 of Lemma 13.2. Since $H \neq 1$, Theorem 17.1, together with (2) of Theorem 11.2, shows that $H^2(G, Z, m) = 0$. The reasoning, used under Case 1 of Lemma 13.2, now proves again that $N \subset G^1$ and that $G/G^1 \cong H/H^1$. Done.

Clearly, Lemma 13.2 and Theorem 17.1 are equivalent statements. It would be highly interesting to have a proof of Theorem 17.1 which does not make use of the Frobenius theorem.

REMARK 17.1. One can easily conclude from the last paragraph of §16 that $H^2(X; G, Z)$ is isomorphic with the intersection of N/N^1 and the center of G/N^1. We see from Theorem 17.1 that, if $H \neq 1$, this intersection is the unit element of G/N^1.

REMARK 17.2. $H^1(X; G, Z) = 0$ by Proposition 6.4 of [1] and $H^2(X; G, Z) = 0$ (if $H \neq 1$) by Theorem 17.1. The periodicity of H makes it unlikely, but not impossible, that $H^p(X; G, Z) = 0$ for all $p > 0$ if $H \neq 1$.

18. **The higher commutator subgroups of G and H.** In the case that H is solvable and $\neq 1$, Theorem 14.1 compares the higher commutator subgroups of G and H. *We assume for this section that H is not solvable.* The properly descending sequence $H = H^0 \supset H^1 \supset H^2 \supset \cdots \supset H^e = H^{e+1} \neq 1$ of the higher commutator subgroups of H may now very well consist of only one term, in which case $e = 0$. Theorem 18.1 states that the sequences of the higher commutator subgroups of G and H have isomorphic factors.

THEOREM 18.1. *$G^i/G^{i+1} \cong H^i/H^{i+1}$ for $i \geq 0$. Furthermore, $N \subset G^e$ and (N, G^e, H^e) is a Frobenius triple.*

Proof. The reasoning, used in the proof of Theorem 14.1, now gives that $G^i/G^{i+1} \cong H^i/H^{i+1}$ for $i = 0, \cdots, e-1$, that $N \subset G^e$ and that (N, G^e, H^e) is a Frobenius triple. Since $H^e \neq 1$, we may apply Lemma 13.2 once more (we are now in Case 2 of that lemma!) and conclude that $H^e/H^{e+1} \cong G^e/G^{e+1}$, i.e., that $G^e = G^{e+1}$. Done.

In connection with Theorem 14.1, it is interesting to observe that N is now not equal to a higher commutator subgroup of G. Namely, since H is not solvable, G cannot be solvable and hence G^e is certainly not abelian. N, however, is abelian.

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